2. Sweet and sour with a hat

teacher guide

Goal of the task

This task has three goals:

- To familiarize students with some features of TI InterActive! that have not been encountered during the introduction session. In particular, students will learn to draw on a Graph, use Stat Plots, learn the syntax for piecewise functions and use sliders in order to obtain parameter variations.

- To test the mathematical skills mentioned before and evaluate the extent to which students have integrated the notion of function and can compose elementary functions in order to obtain, for example, a positive and increasing function for some given interval of the variable.
- To encourage students' creativity.

Target group and required time

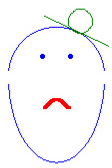
The target group consists of students in 11th grade after first semester, or 10th grade at the end of the academic year. Estimated time to perform:

Main Task Part A: 2 "hours" ("hour" means 1 time period of 50 minutes)
Main Task Part B: 3 "hours"
Additional Task: 2 "hours"
Control Task: 1 "hour"

Preliminary TI InterActive! skills

The students need some basic skills in using TII, in particular the use of Function Windows and Graph Windows. For some parts of the work, it will be useful for students to be able to easily manipulate a Math Box and particularly the functionalities of the Algebra Menu.

Preliminary mathematical skills



Preliminary knowledge in calculus/algebra includes: elementary functions (polynomial function of degree 2, square root function, homographic function), piecewise functions, transformations of the plane applying the graph of f on the graph of $a \cdot f(mx+p)+b$, determination of the parameter(s) appearing in the analytic expression of a function in order that the graph is tangent to a given line (in one of its points).

Eventually, the notion of derivative.

Preliminary knowledge in analytic geometry: perpendicular lines, equation of a circle.

File organization

The unit consists of the following linked TII files:

- Start file: File containing links to all the other files of the group Sweet & Sour except for the different Help and Hint files. Links to these files are situated in Main Task A and B files or Additional Task file or Control Task file.

- Notebook file: Starting file for the student. The student is supposed to insert his or her answers to the various parts of the exercise in this file.
- Main Task file, part A: Introductory file, describing the first part of the task.
- Main Task file, part B: Contains the second and principal part of the task.
- Help 1 and Help 2 files: Contains some technical help to perform some of the required tasks for Main task A.
- Hint 1 file: Contains suggestions that could be useful to the student who does not know how to solve some of the questions from Main task A.
- Hint 2 to Hint 4 files: Contains suggestions that could be useful to the student who does not know how to solve some of the questions from Main task B.
- Additional Task file: Contains an additional task for the brighter or faster student.
- Hint 5 file: Mathematical help to perform the additional task.
- Control Task file: Contains a control question to evaluate the understanding of some of the suggestions made in Hint 4 file.
- Solution file: Contains a possible solution for each of the exercises proposed in Main Tasks A and B, in the Additional Task and in the Control Task file. Ideally, the teacher should delete the link that exists from the Start file to the Solution file before the students start to work!

The hyperlinks only work if the files are installed in the map c:\TII\SweetandSour.

Technical hints

Students start by opening the notebook file in TII. It is recommended that the main task is opened in a separate window so that two versions of TII are activated. By means of a right mouse click in the Windows menu bar the option 'Windows Cascade' offers opportunities for switching easily between the files. However, if the same hyperlink is used twice two versions of the same file will be opened! This may be a source of confusion and therefore requires some attention by the teacher.

As TII only supports absolute hyperlinks and no relative links, the links need to be adjusted after installation of the files, for example on the school network.

Classroom organisation

The students work either individually or in pairs on computers during normal mathematics classes or at any other time. The above files are organized and detailed in such a way that students should be able to work without the presence of their mathematics teacher.

Didactical suggestions

We tried out this unit with students in 11th grade. For technical reasons related to the poor state of the computers in the school, the students worked alone or in groups of two or three at home. Though all of the students were beginners with TI InterActive!, none of them encountered technical difficulties related to the use of the software itself.

On the contrary, none of the students was able to generate the simultaneous movements of the mouth and chin of Sweet and Sour with only one slider bar. Most of them succeeded with two slider bars, but this was not the goal of this part of the task. This probably means that the students did not entirely grasp the concept of function.

A few students even have had problems in choosing an adequate function to represent the mouth and in limiting the function representing the mouth to a certain interval.

The additional task was partially successful in the sense that everybody obtained the linear part of the hat, but the portion of hyperbola did not look as it should have.

2.1 Main task A

intro

The main task consists of two parts, part A and part B, in which you will use, among other features, the:

- graphing tool of TI InterActive!
- stat plot tool
- slider bars
- definition of piecewise functions.

If you do not feel familiar with one of the above options or if you do not know the syntax to be used for piecewise functions, feel free to consult the *help topics* provided in this document or use the TI InterActive! Help ? button.

The mathematical skills you need concern:

- elementary functions and their graphs
- translating and stretching curves
- tangent lines to curves

Some *mathematical hints* are also provided in this document.

Try to be imaginative, and success !

task

a. The neutral face

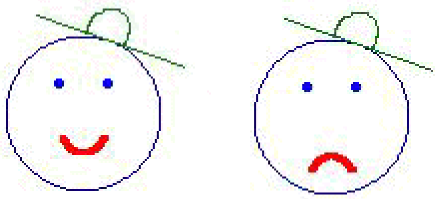
Sketch a small face similar to the one in the following figure using the drawing tool for the circle, mouth and hat, and the stat plot tool for the eyes.

Help 1

b. Sweet changes into Sour

Sketch a face similar to the one in the preceding figure whose mouth shape varies by means of a slider bar, from "smile" to "sad" as illustrated below.

Help 2
Hint 1



2.2 Main task B

task

a. Once more the neutral face

Sketch a small face, similar to the one of the figure below, using:

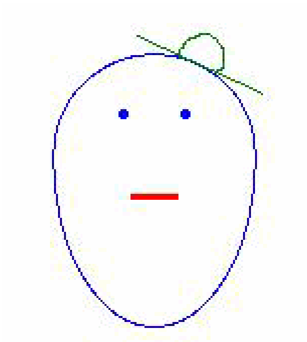
- *two functions* to draw the circle,
- the drawing tool for the hat and the mouth
- the Stat Plot tool for the eyes.

Hint 2

b. Sweet changes into Sour

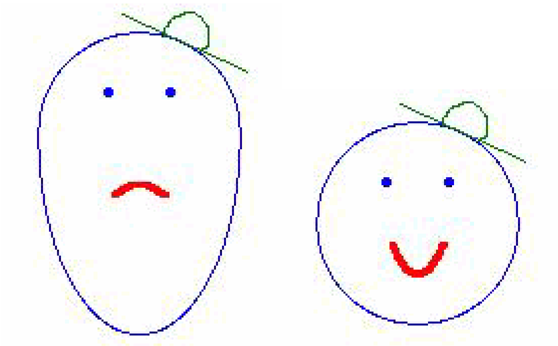
Sketch a face similar to the one in the preceding figure, but this time with a chin becoming longer and longer by means of a slider bar.

Hint 3



c. Using ***only one slider***, sketch Sweet and Sour in such a way that the movements of the chin of Sour correspond simultaneously to a variation of the mouth, as indicated on the following figures.

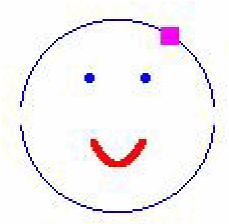
Hint 4



2.3 Control task

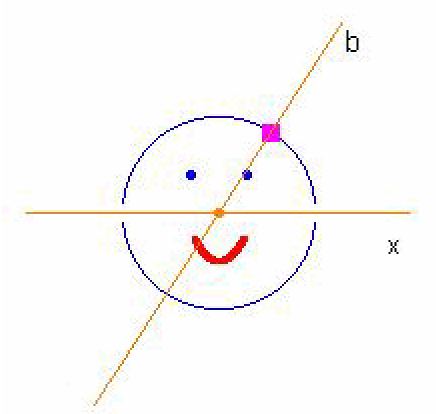
task

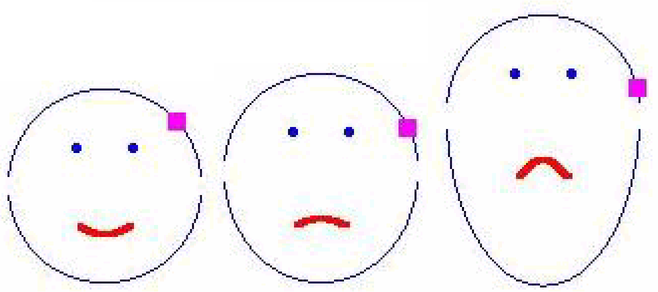
Elvis Pompilio has created a nice tiny hat for our friend ...
Here it is:



a. Sketch the same face as in Task B, delete the green hat (you double-click on the segment then delete it and do the same for the green circle).
Using Stat Plots, create the tiny hat knowing that it is situated on the upper semi-circle of the face and that the angle (x,b) is equal to 1 radian.

b. *Put a parameter* in the hat's coordinates in such a way that the hat moves from its initial angular position of 1 radian to an angular position of a quarter of a radian when you slide the button into the slider bar *controlling the mouth and chin* of Sweet&Sour.



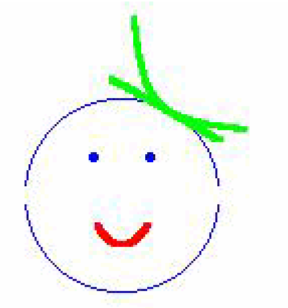


2.4 Additional task

task

Elvis Pompilio has created a nice hat for our friend ...

Here it is.



Sketch the same face as in Task B but delete the green hat (double click on the segment then delete it and do the same for the green circle).
Using well chosen functions, try to create the same type of hat for our fellow!

Hint 5

2.5 Hints

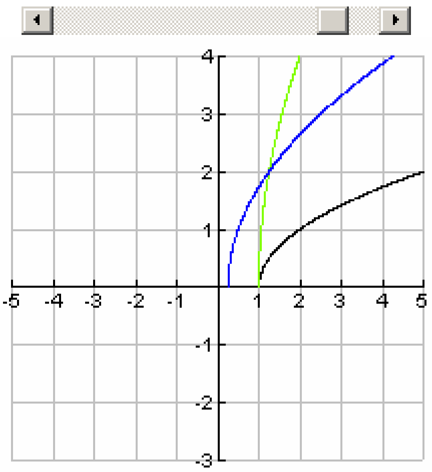
hint 1

Let us consider three functions defined the following way:

define $f(x) = \sqrt{x - 1}$ define $g(x) = k \cdot \sqrt{x - 1}$ define $h(x) = \sqrt{k \cdot x - 1}$

In the figure below, the graph of *f* is the black curve (half a parabola).

Using a slider to change the values of *k*, you can observe the effect of these changes on the green graph of *g* and the blue graph of *h*.



We have $g(x) = k \cdot f(x)$.

From this result we know that the green curve can be obtained by stretching the black one in the direction Oy, axes Ox, factor *k*. The effect of this transformation is just a change in the opening of the half parabola.

We have $h(x) = f(k \cdot x)$.
Hence the blue curve can be obtained by stretching the black one according to direction Ox, axes Oy, factor $\frac{1}{k}$ The effect of this transformation is not only a change in the opening but the intersection point with Ox also changes.

Can you explain what happens when k is equal to zero or equal to one ?

Another way of transforming the graph of *f* would be to add *k* to *f*(*x*) or replace *x* by *x*+*k* inside the algebraïc expression for *f*(*x*) ...

Do you think that this is the type of transformation we need here to change the shape of the mouth of Sweet and Sour ?

hint 2

Drawing a circle with given center and radius

The equation of the circle with center $(0, 0)$ and radius r is $x^2 + y^2 = r^2$.

This circle is the union of the graphs of two functions whose analytic expressions are obtained by solving the preceeding equation with respect to *y*.

$$\text{solve}(x^2 + y^2 = r^2, y)$$

$$y = -\sqrt{r^2 - x^2} \text{ and } r^2 - x^2 \geq 0 \text{ or } y = \sqrt{r^2 - x^2} \text{ and } r^2 - x^2 \geq 0$$

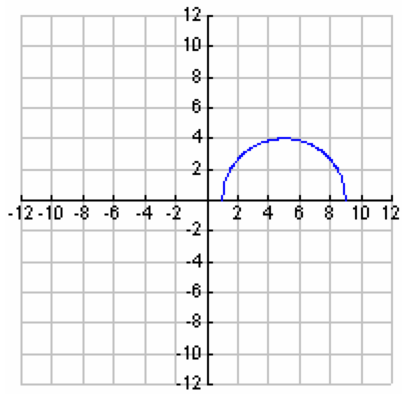
The equation of the circle with center (x_0, y_0) and radius *r* is $(x - x_0)^2 + (y - y_0)^2 = r^2$.
This circle is the union of the graphs of two functions whose analytic expressions are given by

$$\text{y1}(x) := y_0 + \sqrt{r^2 - (x - x_0)^2} \text{ and } \text{y2}(x) := y_0 - \sqrt{r^2 - (x - x_0)^2} .$$

Hint 3.1

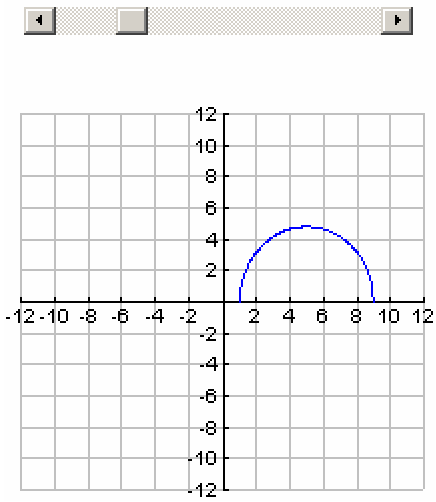
define $f(x) = \sqrt{16 - (5 - x)^2}$

The graph of this function is half a circle with center $(5, 0)$ and radius.



Hint 3.2

Let us introduce a parameter *k* varying from 0.2 to 5 and consider the function $g(x) := k \cdot \sqrt{16 - (x - 5)^2}$ and its graph. Use the slider bar to see the effect on the initial circle when *k* is varied.



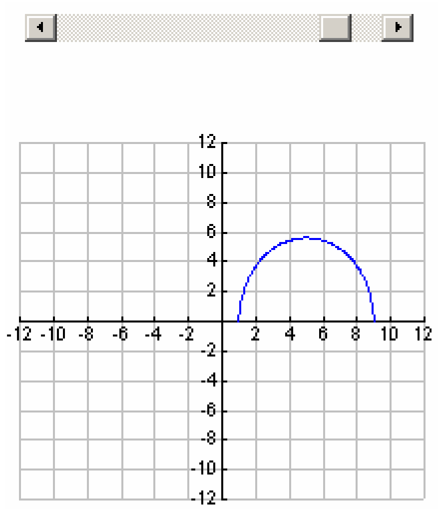
The half circle is stretched in the direction of Oy, the axes of the stretching is Ox and the factor is *k*. The circle is transformed into an ellipse whose intersection points with Ox are fixed.

Remark: The maximum value of *k* is too big with respect to the chosen window giving the impression that the ellipse is getting "open".

Hint 3.3

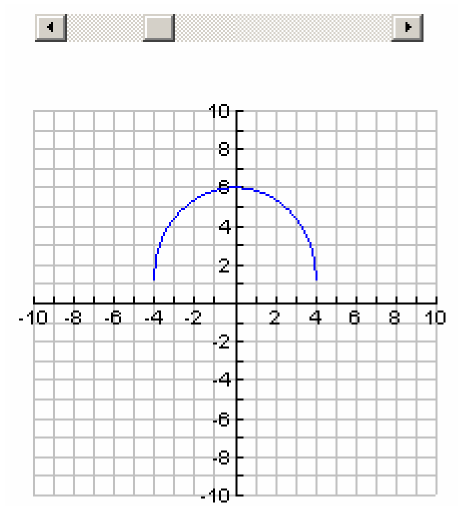
Now observe what is happening if you move the slider bar.

The graph appearing in the figure above is the graph of $k \cdot f(x)$ but the parameter *k* takes negative values (right click on the slider bar, choose open/activate to see the new domain of *k*).



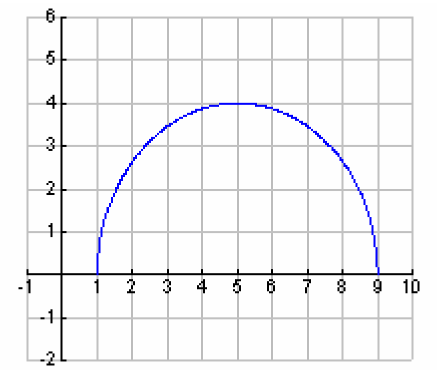
Hint 3.4

What happens if the semi-circle does not cut the Ox axes?
Explain your answer using the following figure and the corresponding slider bar.



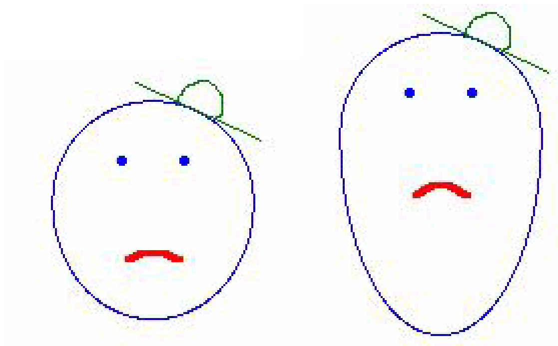
Hint 3.5

And now, explain what you observe in the figure below.
Is the equation of this curve the equation of an ellipse or a circle?



hint 4

Using only one slider would require that the parameter generating the movements of the chin would depend on (be a function of) the parameter generating the variations of the mouth.



To make the preceding figures, the parameter *m* that was chosen for the mouth is strictly negative and decreasing when Sour is sad and getting sadder and sadder.

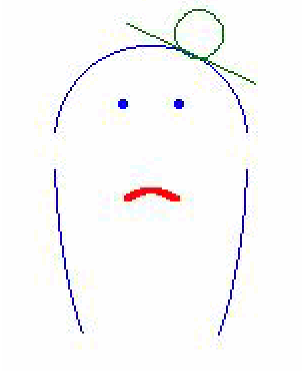
The parameter *ch* generating the movements of the chin had to be greater than 1 and increasing to assure that the chin became longer and longer and to keep the right concavity.

So, in the preceding figures, the problem was to find a function *ch* of *m* that would be positive, greater than 1, increasing when *m* was negative and decreasing and that would not increase too rapidly in order to avoid the chin "opening" itself.

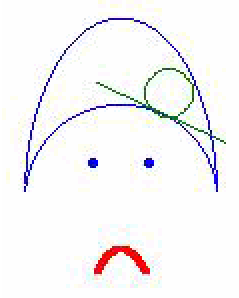
In the following figures, you see some of the misadventures that would have happened to Sour if one had not chosen the right function *ch(m)*:

ch (m) is not greater than 1 .

Here *ch(m)* increases too quickly.



Here *ch(m)* is not positive !

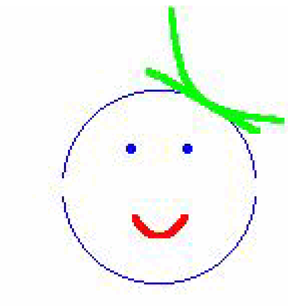


How can you find the function that fits this case ?

First of all, do not look for a sophisticated function. You know a lot of rather elementary functions among which you will find what you are looking for: polynomial functions of degrees one to four, the homographic function, the absolute value and the square root.

Use TI InterActive! to graph the function you have chosen for *ch(m)* (if these are the names of your parameters) and check that this function corresponds to the one you are looking for.

hint 5



a. The hat contains a segment which is tangent to the face at the point T, abscissa 1 and positive ordinate.

It is easy to determine the equation $y = c{\cdot}x + d$ of a line which is tangent to the upper semi-circle at T.

To obtain a segment instead of a line, it suffices to limit the graph of $c{\cdot}x + d$ making use of piecewise functions.

b. The hat contains a portion of curve (equation $y = \frac{n}{x} + q$) that is tangent to the preceding segment at T.

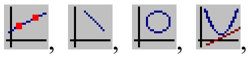
You know different techniques to find n and q in such a way that the graphs of $f(x) = c{\cdot}x + d$ and $g(x) = \frac{n}{x} + q$ are tangent at one given point:

- Express that the two curves intersect each other into the *double* point T or
- Express that the graph of *g(x)* goes through point T and that the slope of the tangent at T is the right one (using derivatives).

2.6 TI InterActive! help

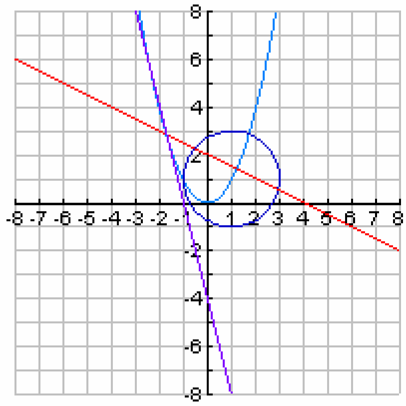
a. Drawing on a graph in a Graph Window

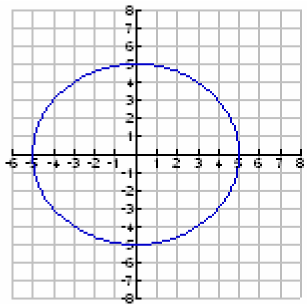
To draw a line using two points, a segment, a circle or a tangent line without needing equations and without affecting any already graphed function use the Draw menu or the corresponding toolbar buttons

 ......

Example:

In one Graph Window produce graphs of the function $f(x) = x^2$, the circle with center (1, 1) radius 2, the line through points (2, 1) and (-2, 3) and the tangent line to the graph of f in its point of abscissa -2.
To see how this was done, double-click on the figure below, then on the circle, line or tangent line
Remark: This was also supposed to be a circle! Do you understand why it is not ?



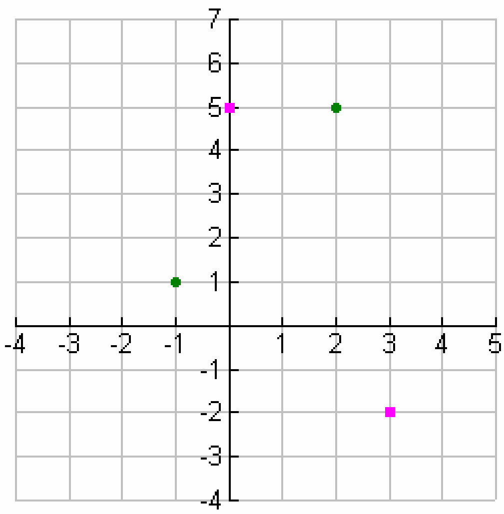


b. Drawing several points in a Graph Window.

Assume that you want to draw the points with coordinates (-1, 1), (2,5), (3,-2),(0,5):

- In a Functions Window click on the Stat Plots tab.
- On the first line insert the abscissas using braces and commas : {-1,2,3,0}.
- On the second line, insert the ordinates in the same order : {1,5,-2,5}.

Again, to see how this was done, double-click on the figure below and click on the Stat Plots tab in the Functions Window.



a. Piecewise functions

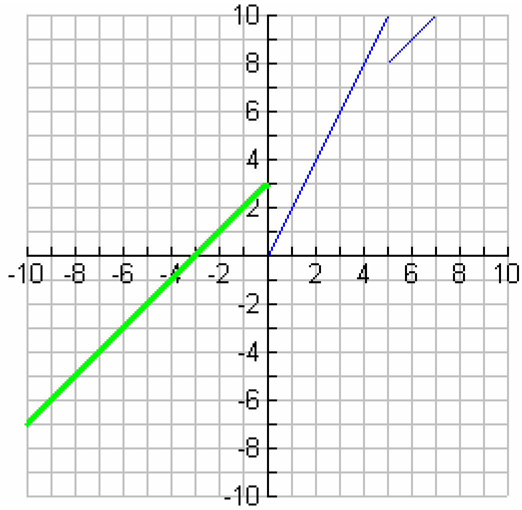
The function $f(x) := \begin{cases} x+3 & x < 0 \text{ or } x > 5 \\ 2x & 0 \le x \text{ and } x \le 5 \end{cases}$ will be defined using the syntax

$\text{when}(x < 0 \text{ or } x > 5, x+3, 2x)$.

The function $f(x) := \{ \; x+3 \quad x > 0$ without any other alternative, will be defined using the syntax

$\text{when}(x < 0, x+3)$.

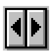
Click on the figure below to see how this was done.



b. Using a slider

Consider the function $f(x) := 2 \cdot x + m$ where m is a parameter taking values between –4 and 10 with step $\Delta m = 2$.

With TI InterActive! it is possible to illustrate the corresponding variations of the graph of f:

First click on the button  of the TII toolbar. Fill in the following fields:

- Variable name: Enter the name of the parameter, in this case, *m*,
- Initial value: Default value given to *m*,
- Choose the minimum and maximum values allowed for *m*,
- Step amount: Step $\pm \;\; \Delta m$ will be used if one clicks on the arrows of the slider,
- Page amount: Step width that will be used to increment or decrement *m* if you click anywhere in a blank area between an arrow and the slider bar.

Below you see the graph of *f.* *U*se the slider to investigate the effect of varying the parameter *m*.