Parameters and functions

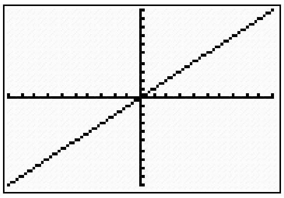
TARGET GROUP
Students upper secondary education – Age: 14-18

TOPIC
Real functions: the effect of changes in a function's parameters to its graph.

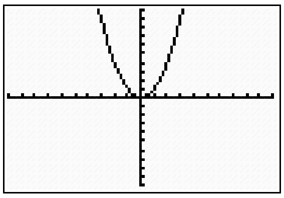
PRIOR MATHEMATICAL KNOWLEDGE
Basic real functions: $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = e^x$ and $f(x) = \sin(x)$

PRIOR CALCULATOR EXPERIENCE
Graphing, Stat editor, Statistical plots, Lists, Matrices

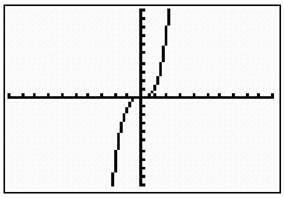
We will start our investigation about the changes in parameters of a function to its graph with the following elementary real functions, all plotted in the standard window (`6:Zstandard`).



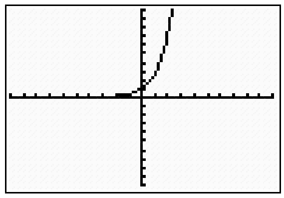
$f(x) = x$



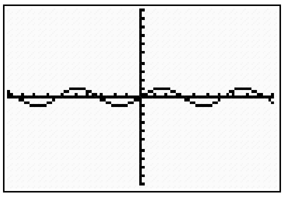
$f(x) = x^2$



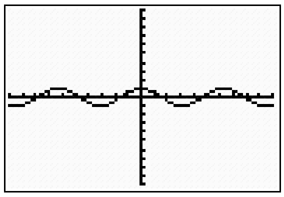
$f(x) = x^3$



$f(x) = e^x$



$f(x) = \sin(x)$

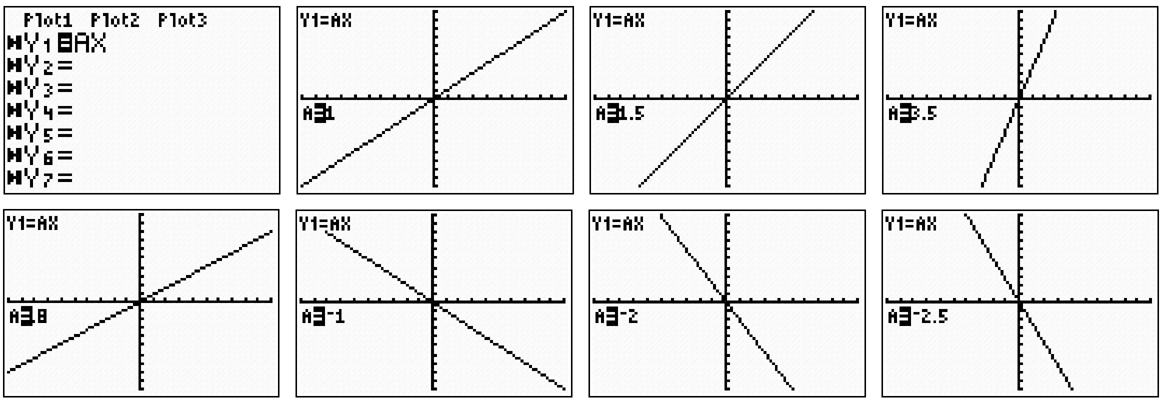


$f(x) = \cos(x)$

With Transformation Graphing we will graphically analyze linear, quadratic, exponential and trigonometric functions. By exploring the graph, students can discover properties of the functions on their own. A next step is to confirm them algebraically, with or without computer algebra. But this is not the meaning of this chapter.

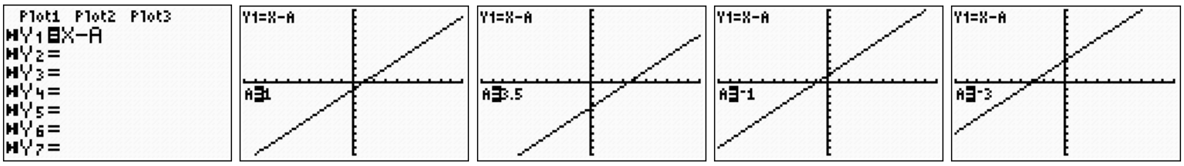
1. Linear functions

a. $y = ax$



The students will observe that the value of a will determine if the function is decreasing ($a < 0$) or increasing ($a > 0$) and that the parameter a corresponds with the slope of the graph. Note that for `A=0` the graph will be equal to the x axis.

b. $y = x - a$



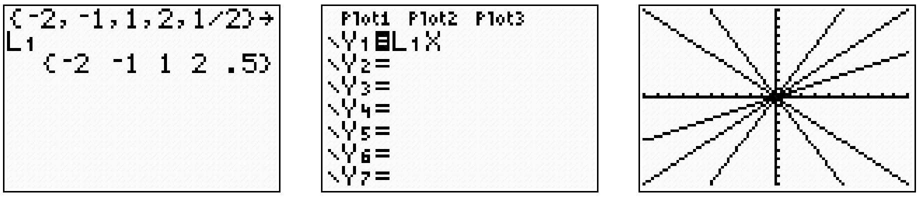
We can conclude that changes in the parameter have no effect on the slope of the line. The transformation $x \mapsto x - a$ move the intersection point of the graph with the y axis upwards ($a < 0$) or downwards ($a > 0$).

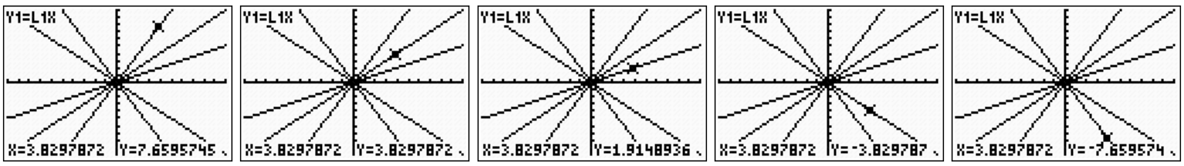
Activity 1

Make a rough sketch of the graph of the function $f(x) = -2x + 3$ and control your graph with your calculator.

A combination of a and b leads to the conclusion that the graph of $f(x) = ax + b$ is a parallel line to the graph of $f(x) = ax$.

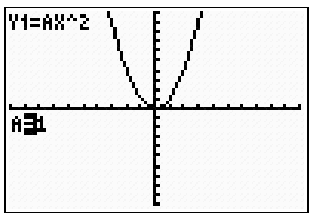
Although Transformation Graphing is a very handy tool to manipulate dynamically the graph of a function it's also interesting to do investigations using lists. An example is shown below.





$y = 2x$ $\quad$ $y = x$ $\quad$ $y = \frac{1}{2}x$ $\quad$ $y = -x$ $\quad$ $y = -2x$

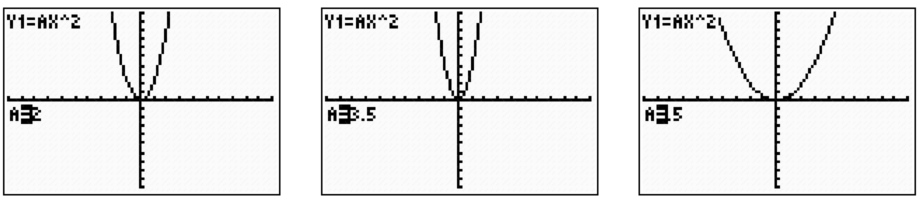
2. Quadratic functions



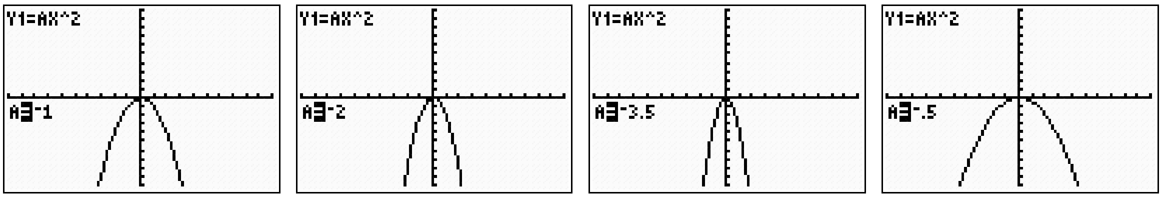
a. $f(x) = ax^2$

The graph of the function $f(x) = x^2$ is a concave up parabola with the origin as the minimum and the y axis, $x = 0$, as the axis of symmetry.

If we change the parameter $a > 0$ we will discover vertical stretches ($a > 1$) and compressions ($a < 1$) of the parabola. The origin stays as the minimum and $x = 0$ is the axis of symmetry.



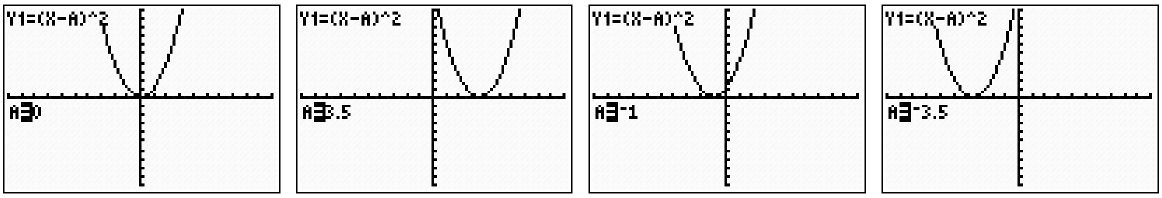
What will happen if a becomes negative?



The graph changes from concave up to concave down and the minimum becomes a maximum. Again we see vertical stretches ($|a|>1$) and compressions ($|a|<1$).

b. $\boldsymbol{f(x)=(x-a)^2}$

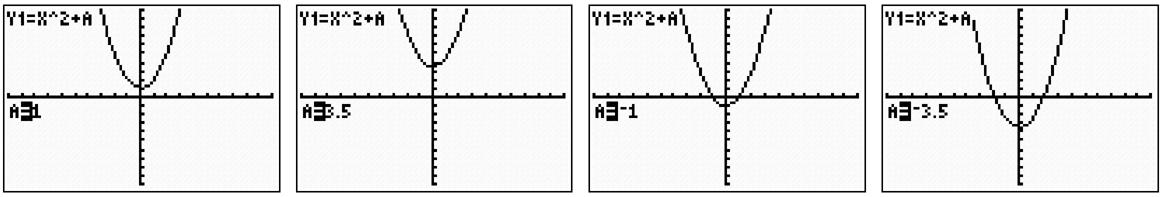
This transformation generates horizontal shifts as shown below.



The minimum will be the vertex $(0,a)$ and $x=a$ the axis of symmetry.

c. $\boldsymbol{f(x)=x^2+a}$

Changing the parameter in this case results into vertical shifts.



$x=0$ is always the axis of symmetry but the minimum is dependent of the parameter a. The vertex $(0,a)$ is the minimum.

d. $\boldsymbol{f(x)=ax^2+bx+c}$

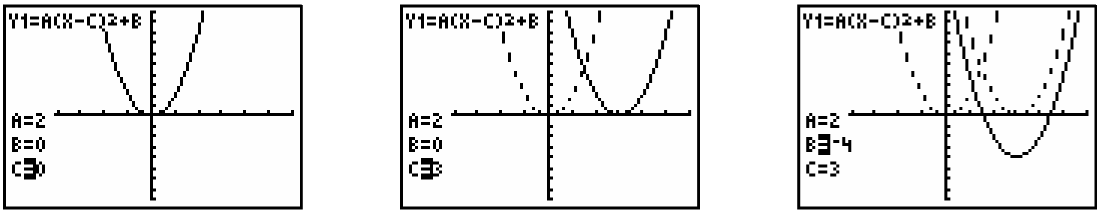
Each quadratic function, $f(x)=ax^2+bx+c$, can be transformed as follows:

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(x^2+2\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)=a\left(x-\left(-\frac{b}{2a}\right)\right)^2-\frac{b^2}{4a}+c\,.$$

This means for the corresponding parabola that the vertex $\left(-\frac{b}{2a},\frac{-b^2+4ac}{4a}\right)$ is the minimum or maximum and $x=-\frac{b}{2a}$ is the axis of symmetry.

An example: $f(x)=2x^2-12x+14=2(x^2-6x+7)=2(x^2-2\cdot 3x+9-2)=2(x-3)^2-4$.

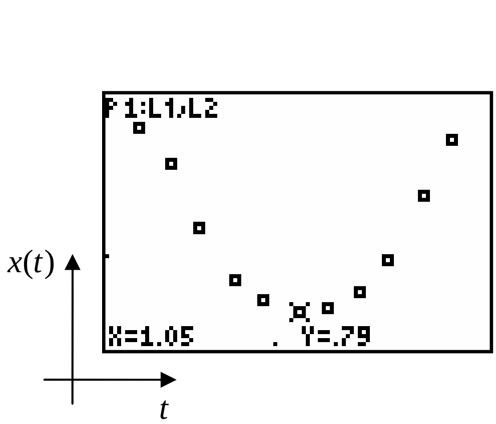
$2x^2$ → $2(x-3)\wedge 2$ → $2(x-3)^2-4$



Activity 2

Determine a quadratic model, $x(t)=a(t-b)^2+c$, for the following data, the bounce of a ball. Use Transformation Graphing to find a value for a .

L1 t	L2 $x(t)$
0.67	1.46
0.75	1.33
0.82	1.09
0.9	0.91
0.97	0.83
1.05	0.79
1.12	0.8
1.2	0.86
1.27	0.98
1.35	1.21
1.42	1.42



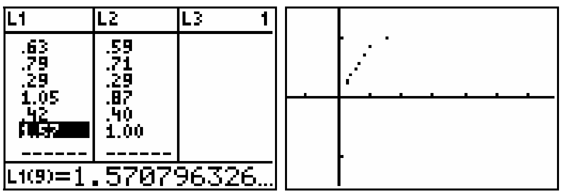
Activity 3

Sketch the graph of the function $f(x)=-(x-2)^3+3$ based on the graph of $f(x)=x^3$. Control your plot with your calculator.

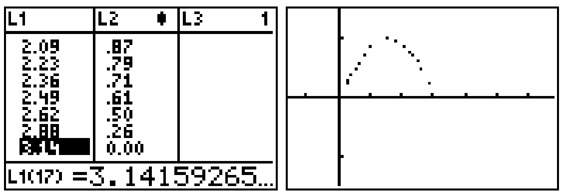
3. Trigonometric functions

Let P be the point where the terminal side of an angle x meets the unit circle. The y-coordinate of this point is equal to the number $\sin(x)$ (and the x-coordinate to $\cos(x)$). We will plot some points $(x,\sin(x))$.

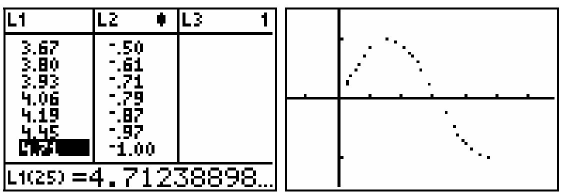
From 0 to $\frac{\pi}{2}$



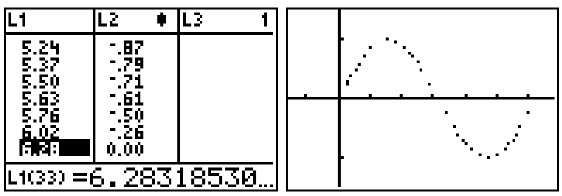
From $\frac{\pi}{2}$ to π



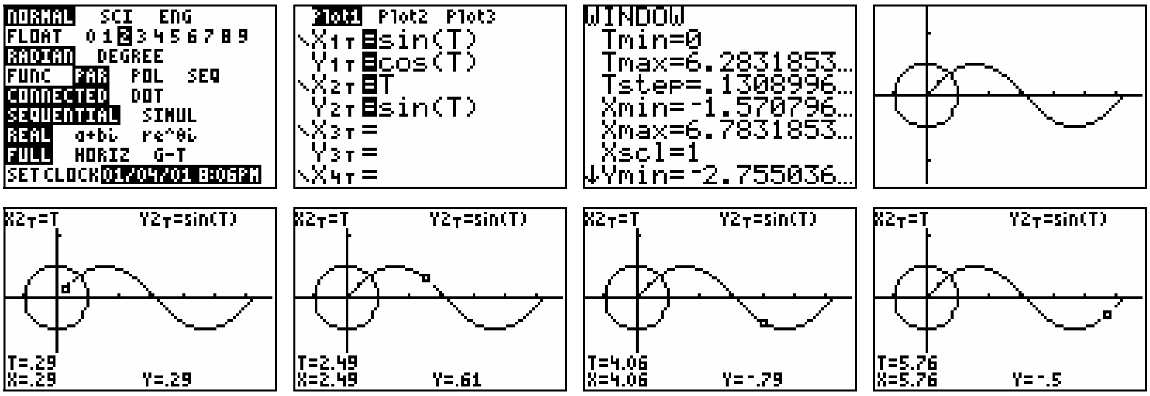
From π to $\frac{3\pi}{2}$



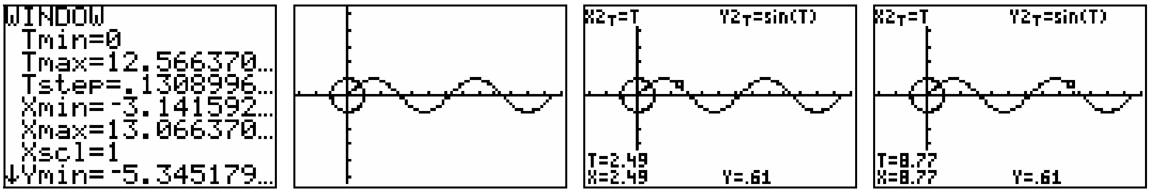
From $\frac{3\pi}{2}$ to 2π



For another visualisation of the sine function we will use parametric functions as follows.

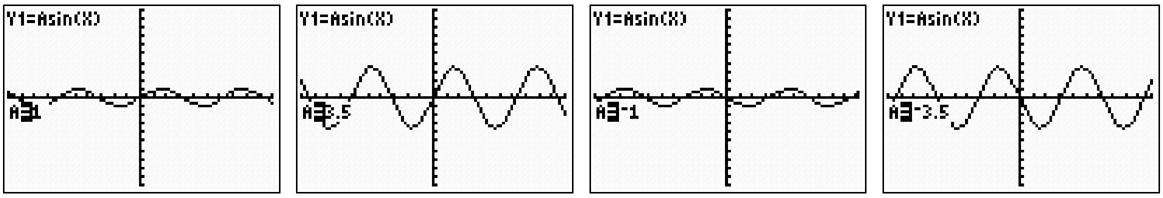


If we extend the range of the parameter from 0 to 4π we can see that $\sin(t) = \sin(t \pm 2\pi)$.
We say that the sine function has a period of 2π .



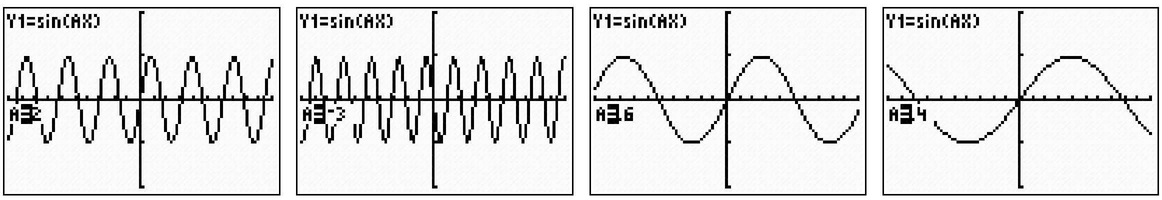
Now we will show some basic transformations of the sine function. The mode has to be equal to `RADIAN`.

a. Vertical stretch or compression - $f(x) = a\sin x$

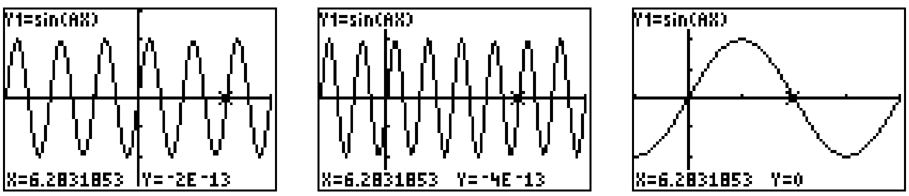


The maximum distance the graph reaches above and below the x-axis is called the amplitude. The amplitude of $f(x) = a\sin(x)$ is equal to $|a|$.

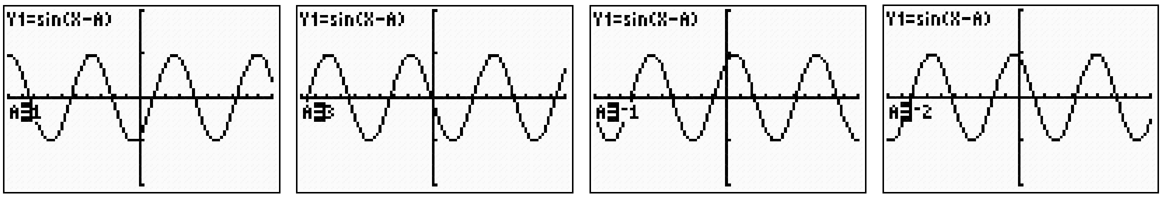
b. Horizontal stretch or compression - $f(x) = \sin(ax)$



All the sine functions we have already seen showed a periodic behaviour and a graph consisting of a series of identical waves. A single wave is called a cycle and its length is called the period. Using `TRACE` leads you to the fact that between 0 and 2π the graph of $f(x) = \sin(ax)$ has |*b*| complete cycles, in other words the period is $\dfrac{2\pi}{|b|}$.



c. Horizontal shift - $f(x) = \sin(x - a)$



In this case the parameter causes a horizontal shift, called the phase shift.

For $f(x) = \sin(x - a)$ the phase shift is equal to a and you can work out that the phase shift for $f(x) = \sin(ax - b)$ is equal to $\frac{c}{|b|}$.

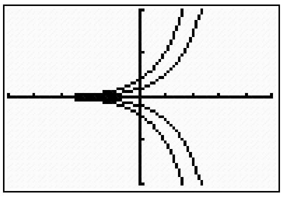
Activity 4

Use Transformation Graphing to model data created by harmonic motion (2.2.4 and 2.2.5)

4. Exponential functions

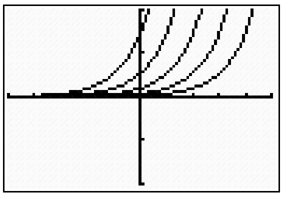
The following screens show some transformations of the graph of $f(x) = e^x$ generated with the list L1={-2,-1,0,1,2} to show the effect of the changes of the parameter a.

$f(x) = ae^x$



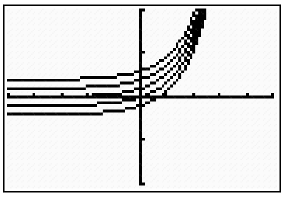
VERTICAL STRETCHES AND COMPRESSIONS

$f(x) = e^{x-a}$



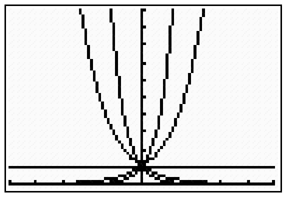
HORIZONTAL SHIFT

$f(x) = e^x + a$



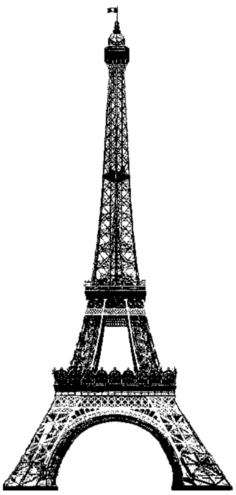
VERTICAL SHIFT

$f(x) = e^{ax}$



HORIZONTAL STRETCHES AND COMPRESSIONS

Activity 5 – The Eiffel Tower



The Eiffel tower is designed according to an exponential model. Stress resistance calculations led Gustave Eiffel (wind, ground support forces etc.) to an exponential frame for the tower, which has become a veritable symbol.

The Eiffel Tower has a height of 300 metres between the ground at the center of the tower and the floor of the upper terrace slightly below the antenna. The floor of the last level is at a height of 280 metres and is approximately 10 metres wide.

The collected data below come from a scaled figure with a height of 15 cm, which is a good scale to work with. We shall define the coordinates of the left foot of the tower as (0,0). Since this value is not actually possible, given that we wish to find a curve that as much as possible resembles an exponential function, we shall only start the measurements from an offset point (5,10).

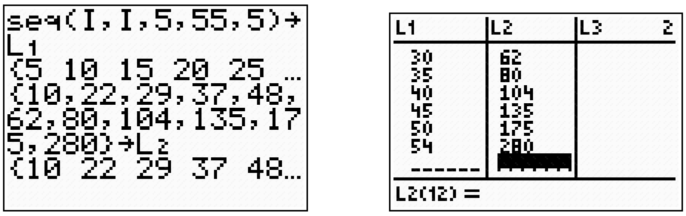
We shall consider the base of the tower to be a square with sides of approximately 115 m.

x	5	10	15	20	25	30	35	40	45	50	54
y	10	22	29	37	48	62	80	104	135	175	280

When modeling the data, certain coordinates may not fit. We shall adopt two attitudes: either suppress the point that is incompatible with the calculations or use a nearby point (at the chosen scale, a pencil line is at least 1/10 mm, which represents 0.2 m in life size).

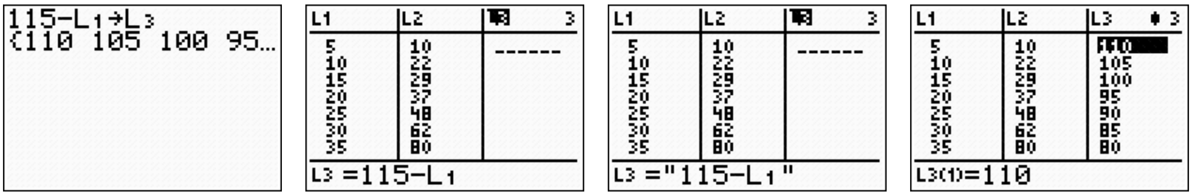
a. Towards an exponential function

We will store the data in the lists L1 and L2. This can be done from the home screen or in the STAT editor.



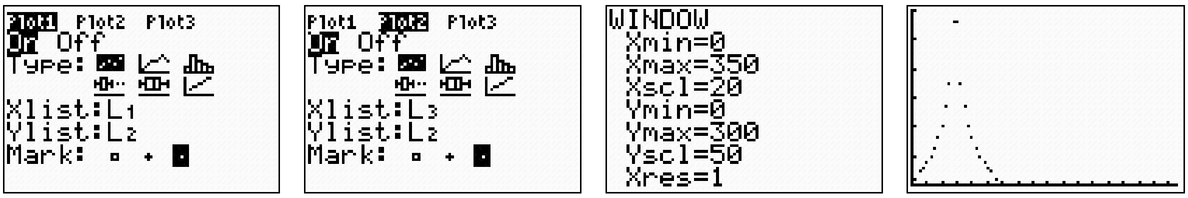
With these data we obtain the right-hand side of the tower. But we want to have two sides.

Five metres from the top, the width of the tower is 110 metres (while at the base, without counting the foundation bulges, it is about 115 metres). It is sufficient to carry out a reflection or axial symmetry in relation to the line $x = 60$.



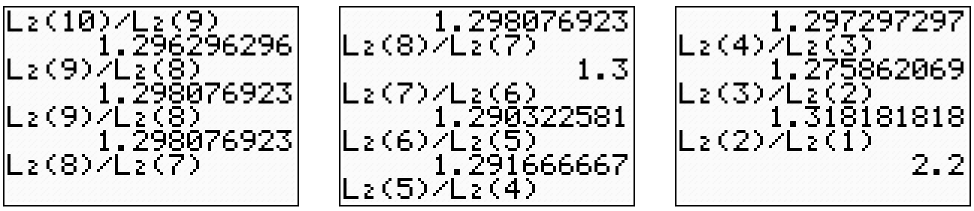
without dynamic link [3]   with dynamic link

Scatter plots of these data result in the following plot.



Note that the ratios of consecutive ordinates (discounting 280 for which the abscissa is not constantly increasing) are "almost" equal. Why "almost" equal?

Because this study is experimental! And because the recorded measurements are approximate … but there is more. Our eyes deceive us to see an exponential frame. In fact, the curve consists of segments of straight lines forming panels between the levels. This is clear from 0 to 50 metres, the floor of the first level.



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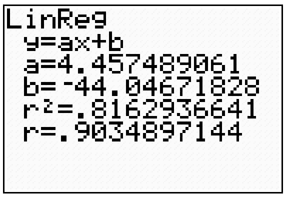
[3] When we write the formula between quotation marks, "115 – L1", the cells of L3 are linked to those of L1, which means that if we change an element of L1, its image in L3 will be recalculated. If the formula is written without changes in L1 will not cause changes in L3.

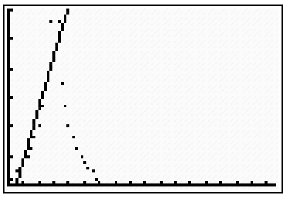
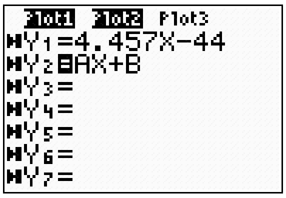
We will now look for functions that best fit the dotted tower.

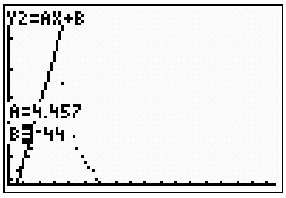
b. Some model tryouts

We will start with a linear model, $y = ax + b$ for the left-hand side, a long distance view.

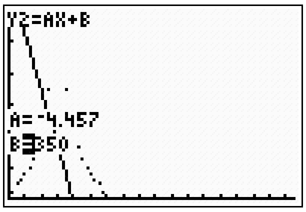
Define $\mathrm{Y1} = 4.457x - 44$ and use Transformation Graphing to attempt to find a straight line for the right-hand side. Start by assigning the values determined for Y1 to the parameters A and B of the function Y2 = AX + B.

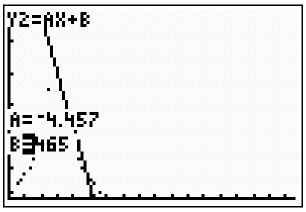


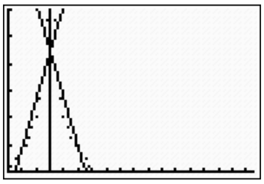


The function we are looking for needs to be decreasing, a line that goes downhill. Therefore the slope, A, of the line has to be negative. B has to be positive and as you see on the graph below more than 350. It is easy to see that A has to be equal to -4.457 and some exploration with Transformation Graphing shows that for B we can accept a value between 450, too small, and 470, too high.





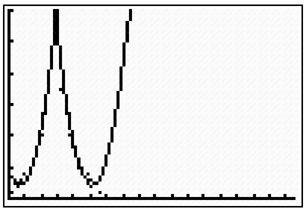
A LITTLE BIT OF MATH



The line $\mathrm{Y2} = a_2x + b_2$ is symmetrical to the line $\mathrm{Y1} = a_1x + b_1$ in relation to the line $x = 5.75$. Therefore the two slopes needs to be opposite: $a_2 = -a_1$.

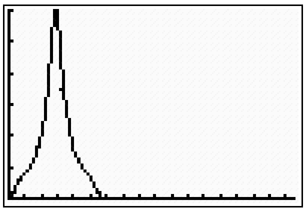
Because Y2(57.5) = Y1(57.5) ($\Leftrightarrow 4.457 \times 57.5 - 44 = -4.457 \times 57.5 + b_2$) we get that $b_2 \approx 468$.

Three other attempts to model the Eiffel Tower. Only the regression results for the left-hand side are shown.



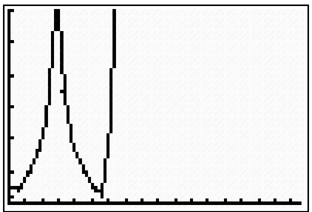
Quadratic model

$ax^2 + bx + c$



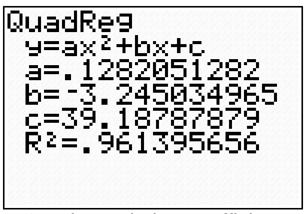
Cubic model

$ax^3 + bx^2 + cx + d$

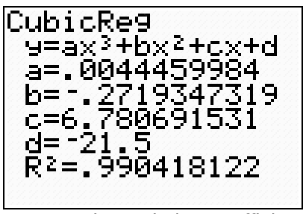


4th degree model

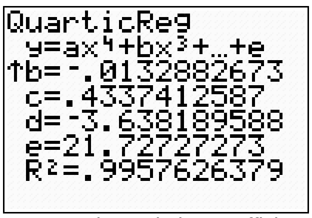
$ax^4 + bx^3 + cx^2 + dx + e$



Good correlation coefficient



Very good correlation coefficient



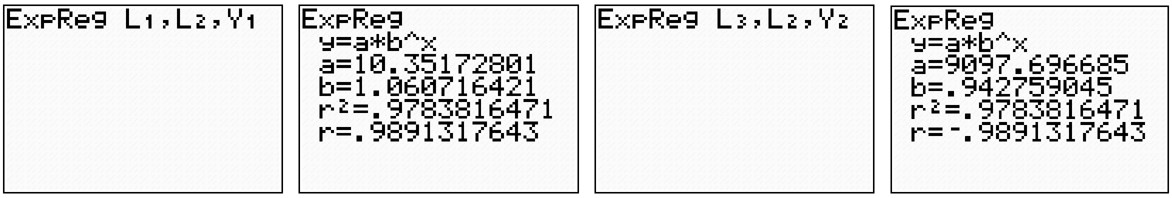
Very good correlation coefficient

Cox said, *"All models are false, but some are useful."* The correlation coefficients above show that several models are suitable to model the Eiffel Tower, but we know that the model chosen by Gustave Eiffel is exponential.

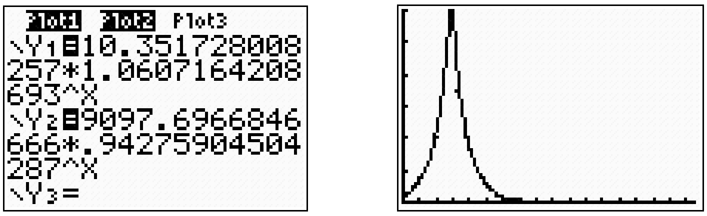
c. The exponential model

Why do we choose an exponential model?

Our eyes are trained to do this but pupils have to observe for themselves that if a function increases more and more (or decreases less and less) it describes probably an exponential an exponential process. In a situation of increasing less and less you should think about a logarithm function (chemistry).



The left-hand side The right-hand side

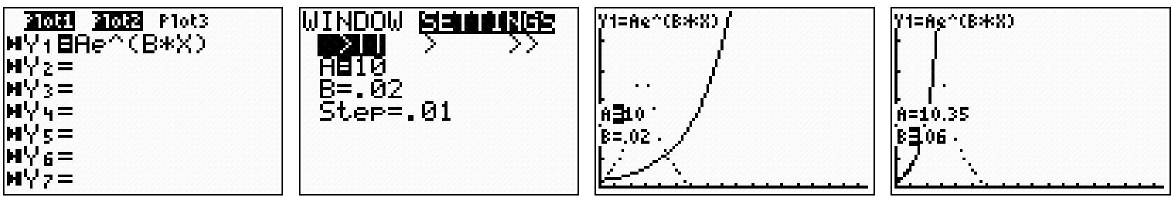


CAUTION

The accuracy has a very significant effect on its results. In some cases this influence is not important. But in case of a regression, fixing the decimal mode to one or two decimals can produce some strange results.

The function $f(x) = 10.35 \times 1.06^x$ gives as a good exponential model for the left-hand side of the tower. This function can also be written in the following form $f(x) = Ae^{Bx}$.

We will use Transformation Graphing to determine the parameters A and B. We will start with the values $A = 10$, $B = 0.02$ and a step size of 0.01.



We can find the required values for A and B as follows algebraically.

- From the equation $a \cdot b^x = a \cdot e^{x \ln b}$,
- We know that the graph has to pass through the points (25,48) and (54,280).

  This gives us the following system of equations: $\begin{cases} 48 = a \cdot e^{25b} \\ 280 = a \cdot e^{54b} \end{cases} \Leftrightarrow \begin{cases} a \approx 10.49 \\ b \approx 0.06 \end{cases}$.

d. Some decorations

Note that in reality, the bottom part (50 m) consists of straight-line segments. Therefore we will calculate, using regression, linear models for the following data.

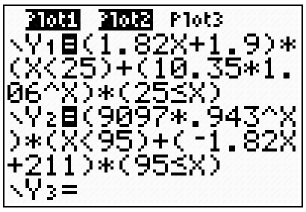
x	5	10	15	20	25
y	10	22	29	37	48

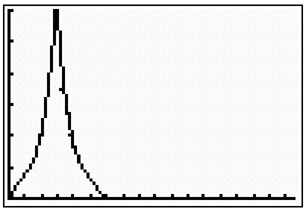
$$y_1 = 1.82x + 1.9$$

x	110	105	100	95	90
y	10	22	29	37	48

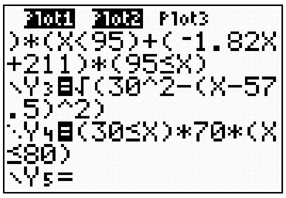
$$y_2 = -1.82x + 211.$$

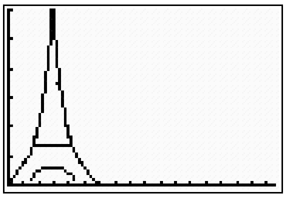
The left-hand (or right-hand) curve consists out of two parts: a straight segment followed by an exponential curve. The TI-84 Plus can plot these using piecewise functions.

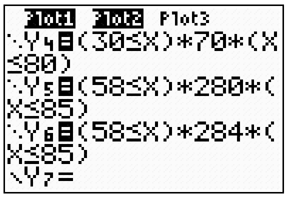


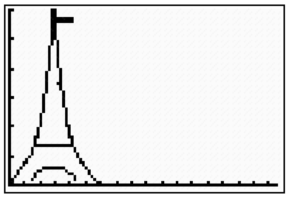


Y3 defines the lower semicircle of the tower and Y4 [4] the floor of the first level.
And add a little flag…









[4] Notice the choice of plot style for Y4, Y5 and Y6.

AN ILLUSTRATION BY THE HAND OF GUSTAVE EIFFEL