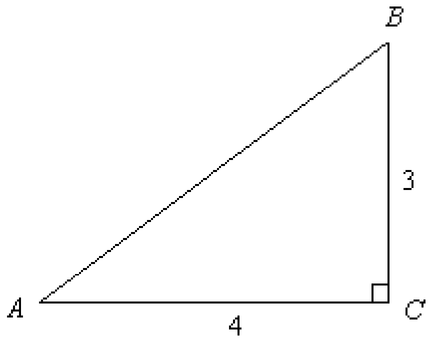
# Danish Baccalaureate May 2000 - Mathematics, Written Exam

## 1. A-level - 2 hours - Total: 50 points
## Paper and pencil only

**Question 1** Expand $\left(2x+y\right)^2-4x(x+y)$.
(2 points)

**Question 2** Solve the inequality $x^2+3x+2<0$.
(4 points)

**Question 3** In the triangle *ABC*, $\angle C=90°$. The sides are 3 and 4 (see fig.)
(4 points)




Calculate $\sin(A)+\cos(A)+\tan(A)$.

**Question 4** Calculate the integral $\int_0^1 3x^5dx$.
(3 points)

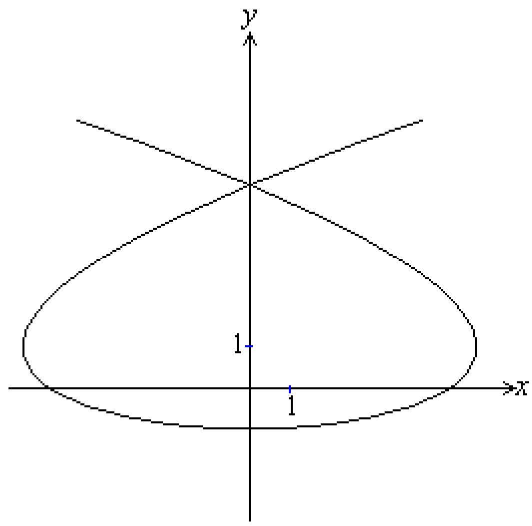
**Question 5** A sphere is given by the equation $x^2+y^2+z^2-2x+4y-4z=16$.
(6 points)
Find the radius of the sphere and the coordinates of the centre.
Given a point *P*(4, 2, 2) on the sphere, find the equation of the plane tangential to the sphere at *P*.

**Question 6** A random variable *X* is normally distributed with mean 15 and standard deviation 2.
(4 points)
Find the probability $P(14\le X\le 18)$.

<u>Note</u>: Normal distribution paper permitted

**Question 7** (6 points)

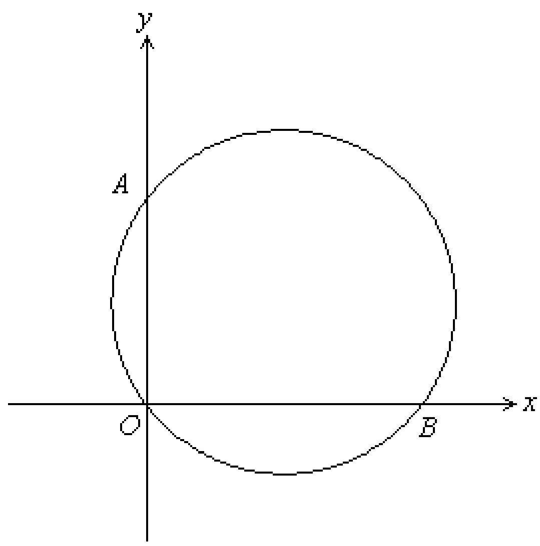
The movement of a point $P(x, y)$ as a function of time $t$ is given by the parametric equations $\begin{cases} x = t^3 - 6t \\ y = t^2 - 1 \end{cases}$ $t \in \mathbb{R}$.




Calculate the coordinates of the intersection points with the $y$-axis.
Calculate the coordinates of the points on the path where the velocity vector is parallel to one of the axes.

**Question 8** (5 points)

The figure below shows a circle with centre in the first quadrant.




The circle passes through the origin $O(0, 0)$ and intersects the axes at $A(0, 3)$ and $B(4, 0)$. Find the equation of the tangent line at $B$.

**Question 9** (5 points)

The size of a population is given by the function $N(t)$ , where N(t) is the number of individuals at time $t$ hours. Given that $N(t)$ is a quadratic function with $N'(t) = 4t + 3$ and $N(0) = 800$.

Find the size of the population at the time $t = 10$ .

**Question 10** (4 points)

Calculate the limits:

$$\lim_{x\to 0} \frac{1-x^2}{4-2x} \quad \text{and} \quad \lim_{x\to 0} \frac{(2+3x)^2-4}{6x}.$$

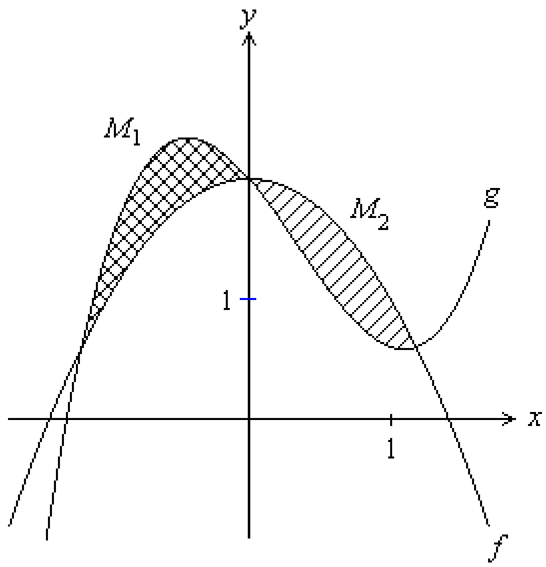
**Question 11** (7 points)

The functions $f$ and $g$ are given by

$f(x) = -x^2+2$ and $g(x) = x^3 - x^2 - bx + 2$ , where $b$ is positive.

The graphs of $f$ and $g$ define two regions $M_1$ and $M_2$ (see fig.).

Show that $M_1$ and $M_2$ have the same area.




## 2. A-level - 4 hours - total: 100 points

**Books, tables of formulae, graphic calculators, corrected homework etc. permitted**

**Questions 7a and 7b are optional ( pupils must choose one)**

**CAS version of exam questions no. 3, 4, 6, and 7a . Pupils taking the CAS version may also use TI-89/92, Computer with MathCad etc.**

**Question 1 (10% , no special CAS version)**

In an orthonormal coordinate system, two vectors are given by $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} t \\ 1 \end{pmatrix}$.

Calculate $t$, when $\det(\vec{a},\vec{b}) = -10$ .

For this value of $t$, calculate the angel between $\vec{a}$ and $\vec{b}$ .

**Question 2 (15% , no special CAS version)**

In triangle $ABC$, $\angle A = 20°, \angle B = 30°$ and $|AC| = 8$.

Calculate the length of the sides $AB$ and $BC$.
Calculate the length of the altitude from $C$.
Calculate the length of the median line from $C$.

**Question 3 (15%) [CAS version in brackets]**

In an orthonormal coordinate system, a line $l$ is given by the parametric equations:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Find an equation for the plane which passes through the line $l$ and the point $Q$ (4, -2, 5) .

A sphere $K$ is given by the equation

$(x-4)^2 + (y+2)^2 + (z-5)^2 = 11$ . [ $(x-a)^2 + (y+2)^2 + (z-5)^2 = 11$, where $a > 2$ ]

The sphere $K$ and the line $l$ have exactly one common point $P$.

Find the coordinates of the point $P$. [Find $a$ and the coordinates of the point $P$ ]

Find an equation of the plane which is tangential to the sphere at the point $P$.

**Question 4 (25%) [CAS version in brackets]**

Consider the real function $f$ defined by
$f(x) = x - 2 \cdot \sqrt{x}$ , $x \geq 0$.

Determine the zeros of $f$ and the intervals where $f$ is increasing/decreasing.

Draw the graph of $f$, and determine the range.
A region $M$ in the fourth quadrant is bounded by the graph of the function $f$ and the $x$-axis.

Calculate, using anti-derivatives, the area of $M$.

Use integration to calculate the volume of the solid of revolution generated by the rotation of $M$ around the $x$-axis.

[Consider a family of real functions
$f_a(x) = x - a \cdot \sqrt{x}$ , $x \geq 0$,
where the parameter $a$ is a positive number.

Determine the zeros of $f_2$ and the intervals where $f_2$ is decreasing / increasing.

Draw the graph of $f_2$ and determine the range.
[A region $M_a$ in the fourth quadrant is bounded by the graph of $f_a$ and the $x$-axis.
Given that the area of $M_a$ is $A_a$ , and the volume of the solid of revolution generated by the rotation of $M_a$ around the $x$-axis is $V_a$.

Calculate $A_a$ and $V_a$.
Calculate $a$, such that $V_a = \pi \cdot A_a$ .]

**Question 5 (10 % , no special CAS version)**

A manufacturer produces components for computers. The components are produced by the machines *A*, *B*, and *C*, making 60%, 25%, and 15%, respectively, of the total production. Experience shows that 20% of the components produced by *A*, 15% of the components produced by *B*, and 5% of the components produced by *C*, are defective.

Calculate the probability that a component, chosen at random, is defective. A component, chosen at random, appears to be defective. Calculate the probability that it was produced by machine *C*.

**Question 6 (15%) [Extra CAS question in brackets]**

Given that a function *f*, is a solution of the differential equation $\frac{dy}{dx} = \frac{y}{\ln(y)}(x+2),\ y > 1$,

and that the graph of $f$ passes through the point $P$(2,e).
Determine an equation of the line which is a tangent to the graph of $f$ at the point $P$.
Determine the solution $f$, including the domain of $f$.

[Extra CAS: Show that the equation $f(x) = f'(x)$ has no solutions. ]

**Choose question 7a or 7b**

**Question 7a (10%) [CAS version in brackets]**

Consider the function

$f(x) = \frac{x^3 + 4x^2 - 4x - 17}{x^2 + x - 12}$. [ $f(x) = \frac{x^2 + 3x - 4}{x+1} + \frac{2x^3 - 4x^2 + 7x - 12}{x^2 - 4}$. ]

Find the domain of $f$.
Find the equation of the asymptotes of *f*.

**Question 7b (10% , no special CAS version)**

For the Galapagos Islands, a good approximation for the number of species of terrestrial plants on any island can be determined as a function $N(x) = b \cdot x^a$ of the area *x*, in square miles.

Given that $N(15) = 68$ and $N(174) = 149$, calculate *a* and *b*.

Consider two islands, where the area of one island is 2.5 times the area of the other island.
Calculate the ratio of the number of terrestrial species on these two islands.

## 3. Solution (with TI89/92Plus/Voyage 200) of Danish Baccalaureate May 2000 – Mathematics, Written Exam A-level

<u>Format of the solution</u>

Left column: Text of the question, and text of the solution, not a model for pupils, but a short version.

Right column: Category of question and calculator screens.

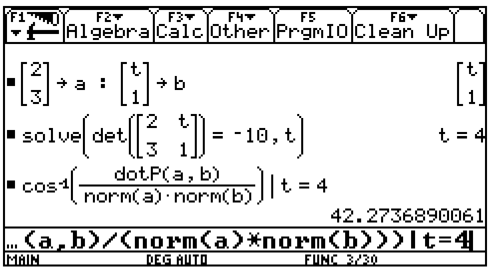
The left and right columns are to be read together.

**Question 1 (10% , no special CAS version)** **Category C0 (CAS-neutral)**

In an orthonormal coordinate system, two vectors are given by $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} t \\ 1 \end{pmatrix}$.

Calculate *t*, when $\det(\vec{a}, \vec{b}) = -10$.

For this value of *t*, calculate the angle between $\vec{a}$ and $\vec{b}$.



**Answer to question 1**

We store the two given vectors.

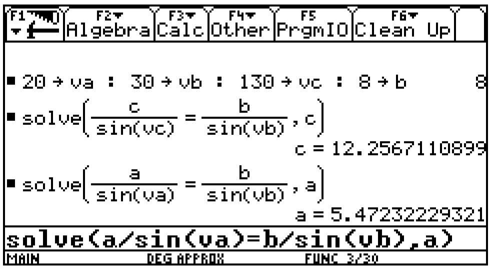
The determinant is -10, when $t = 4$.

The angle between the two vectors (42.27º) is found using the formula $\cos\left(\angle(\vec{a},\vec{b})\right) = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$.

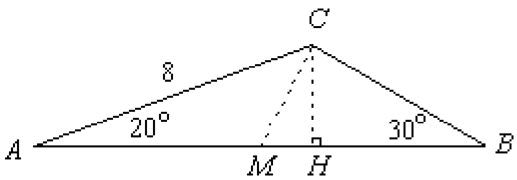
**Question 2 (15% , no special CAS version)** **Category C0 (CAS-neutral)**

In triangle $ABC$, $\angle A = 20°$, $\angle B = 30°$ and $|AC| = 8$.

Calculate the length of the sides $AB$ and $BC$.
Calculate the length of the altitude from $C$.
Calculate the length of the median line from $C$.

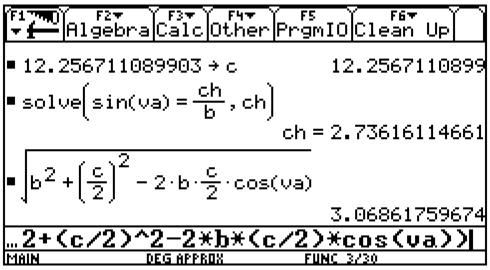



**Answer to question 2**




First we store the given sides and angles. We denote the sides by $a$, $b$, and $c$, and the angles by $va$, $vb$, and $vc$. Using the sine relation we find that $AB \approx 12.26$ and $AC \approx 5.47$.

The value for $AB$ is copied and stored in c .
The altitude $CH \approx 2.74$ is found from triangle $ACH$.
The median segment $CM \approx 3.07$ is found from triangle $ACM$.

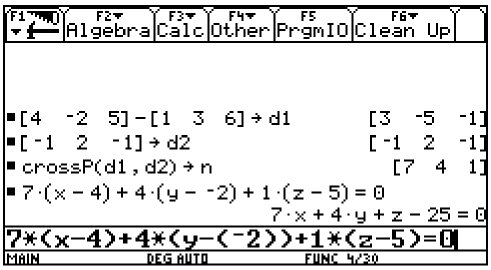



**Question 3 (15%) [CAS version]** **Category C3**

In an orthonormal coordinate system, a line $l$ is given by the equations: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$. Find an equation for the plane $\alpha$ through $l$ and $Q$ (4, -2, 5) .

A sphere $K$ is given by the equation $(x - a)^2 + (y+2)^2 + (z - 5)^2 = 11$, where $a>2$ .

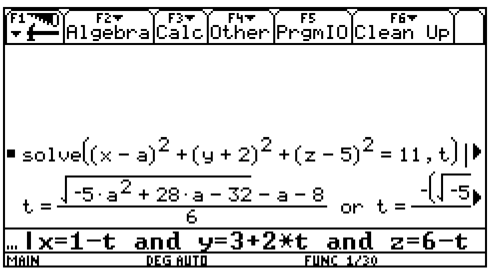
The sphere $K$ and the line $l$ have exactly one common point, $P$.Find $a$ and the coordinates of the point $P$.
Find an equation for the plane tangential to the sphere at the point $P$ .



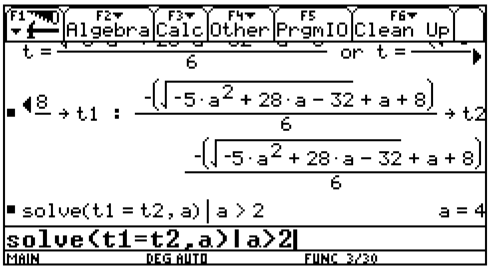

**Answer to question 3**

The plane $\alpha$ is determined by the direction vectors $\vec{d}_1 = (4,-2,5)-(1,3,6) = (3,-5,-1)$ and $\vec{d}_2 = (-1,2,-1)$.

Thus a normal vector is $\vec{n} = \vec{d}_1 \times \vec{d}_2 = (7,4,1)$, and an equation for $\alpha$ is: $7(x-4)+4(y-(-2))+1(z-5)=0$, or $7x+4y+z-25=0$.

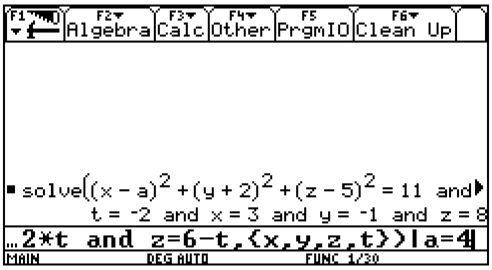


We now solve the equation of the sphere for *t*, using the three equations for the line as conditions.
The two answers for *t* are copied and stored in $t_1$ and $t_2$.
Then equation $t_1 = t_2$ is solved for the parameter *a*.
Finally we solve the four equations, one for the sphere and three for the line, when *a* = 4. Note that there is only one answer. When the sphere and the plane have only one point *P* in common: *a* = 4 and *P* (3,-1,8).



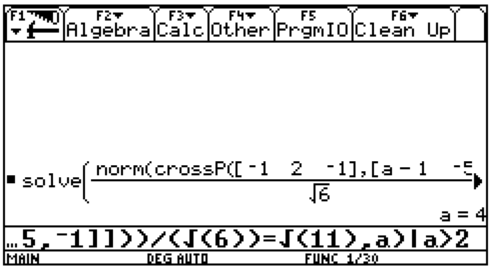

**Alternative method:**
The centre of the sphere is *C* (*a*,-2,5).

With $P_0(1,3,6)$ on the line *l* , $\overrightarrow{P_0C} = \begin{pmatrix} a-1 \\ -5 \\ -1 \end{pmatrix}$,




and the distance from the centre *C* of the sphere to the line *l* is $\text{dist}(C,l) = \dfrac{\left| \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} a-1 \\ -5 \\ -1 \end{pmatrix} \right|}{\sqrt{6}}$, where the denominator is the norm of *l* 's direction vector :

$\sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$ .




The condition for *a* is that dist(*C*, *l* ) equals $\sqrt{11}$ which is the radius of the sphere. Once again we end up with *a* = 4, which is the same result as above.

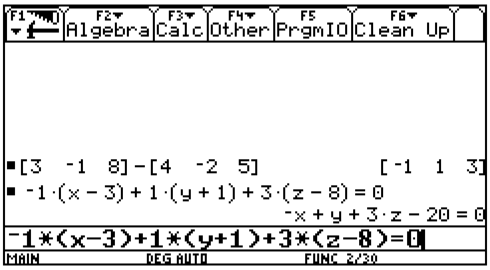
This subquestion can be solved in several other ways.

The normal vector of the tangent plane at *P* is

$\begin{pmatrix} 3-4 \\ -1-(-2) \\ 8-5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$, and the equation

$-1(x-3) + 1(y-(-1)) + 3(z-8) = 0$ , or

$-x + y + 3z - 20 = 0$ .



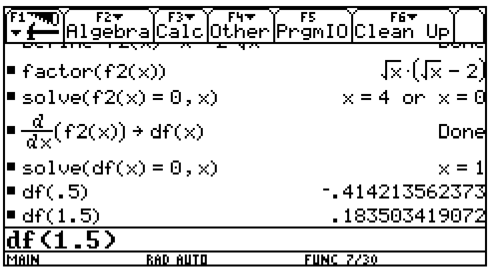

**Question 4 (25%) [CAS version]** **Category C3**

Consider a family of real functions

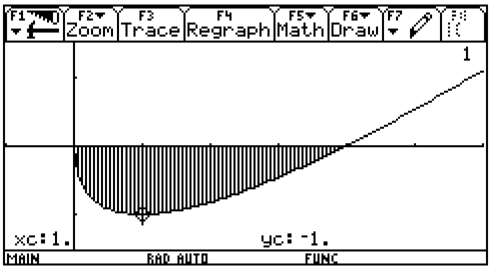
$f_a(x) = x - a \cdot \sqrt{x}$ , $x \geq 0$ , where *a* is a positive parameter.

Determine the zeros of $f_2$ , and the intervals where $f_2$ is decreasing or increasing.

Draw the graph of $f_2$ , and determine the range of $f_2$



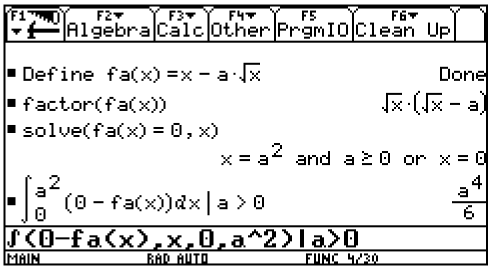
A region $M_a$ in the fourth quadrant is bounded by the graph of $f_a$ and the *x*-axis. The area of $M_a$ is $A_a$, and the volume of the solid of revolution generated by the rotation of $M_a$ around the *x*-axis is $V_a$.




Calculate $A_a$ and $V_a$.
Calculate *a*, such that $V_a = \pi \cdot A_a$.

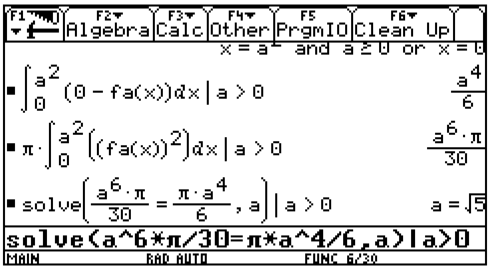
**Answer to question 4**

The function $f_2$ is defined (barely visible on the screen shown in the right column). By factorising $f_2$ or by solving the equation $f_2(x) = 0$ we find the zeros of $f_2$, namely $x = 0$ or $x = 4$.




We then differentiate $f_2$, and denote the derivative by $df(x)$. The derivative is zero when $x = 1$.

We therefore calculate the values of the derivative at (e.g.) $x = 0.5$ and at $x = 1.5$, and find that to the left of $x = 1$ the derivative is negative and to the right it is positive.




Thus $f_2$ is decreasing when $0 \le x \le 1$ and increasing when $x \ge 1$.

The minimum is $f_2(1) = -1$. The factorisation of $f_2$ shows that $f_2(x) \to \infty$ as $x \to \infty$.

Since $f_2$ is continuous, we can now conclude that the range of $f_2$ is the interval $[-1; \infty[$ .

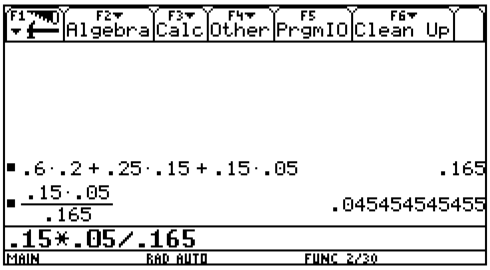
The factorisation of $f_a(x)$ shows that the graph of $f_a$ has the same overall shape for all $a > 0$, and the region $M_a$ is bounded by $x = 0$ and $x = a^2$.

The area $A_a$ and the volume $V_a$ are found by integration over the interval $[0; a^2]$. The answers are:

$A_a = \dfrac{a^4}{6}$ and $V_a = \pi \cdot \dfrac{a^6}{30}$. The volume is found by multiplying the area by $\pi$ when $a = \sqrt{5}$.

**Question 5 (10 % , no special CAS version)** **Category C0 (CAS-neutral)**

A manufacturer produces components for computers. The components are produced by machines *A*, *B*, and *C*, making 60%, 25%, and 15%, respectively, of the total production.




Experience shows that 20% of the components produced by *A*, 15% of the components produced by *B*, and 5% of the components produced by *C*, are defective.

Calculate the probability that a component, chosen at random, is defective.A component, chosen at random, appears to be defective. Calculate the probability that it was produced by machine *C* .

**Answer to question 5 ( see screen)**

**Question 6 (15%)** **Categories C0, C1, C2**

**[Extra CAS question in brackets]**

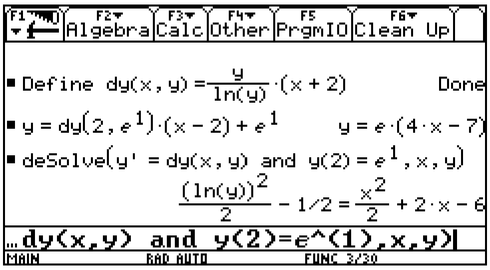
A function $f$ is a solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{\ln(y)}(x+2),\ \ y > 1,$$

and the graph of $f$ passes through the point $P(2, \mathrm{e})$.

Determine an equation of the line tangent to the graph of $f$ at *P*. Determine the solution $f$, including the domain of $f$.

[Extra CAS: Show that the equation $f(x) = f'(x)$ has no solutions.]

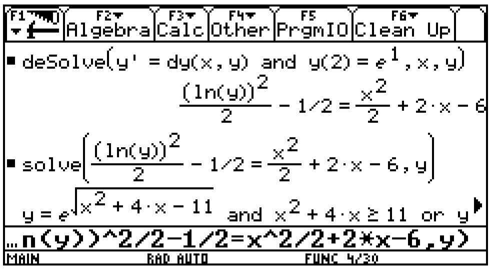



**Answer to question 6**

The tangent line at $P(2,\mathrm{e})$ can be found before solving the differential equation.
The equation is $y = 4\mathrm{e}x - 7\mathrm{e}$.

Solving the differential equation using deSolve gives us an implicit solution.
When we solve for *y* we get two answers:

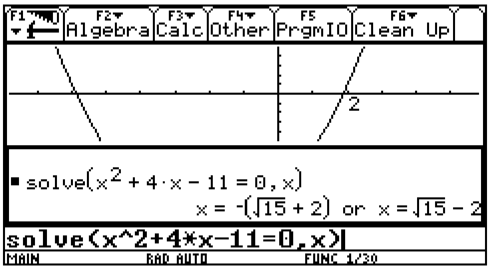
$y = \mathrm{e}^{\pm\sqrt{x^2+4x-11}}$, with the condition that $x^2 + 4x > 11$.




Since the graph passes through the point *P*(2, e), and the range satisfies *y* > 1, we cannot use $y = \mathrm{e}^{-\sqrt{x^2+4x-11}}$,

To find the domain we study the graph and the sign of $x^2 + 4x - 11$. We find that the sign is positive when

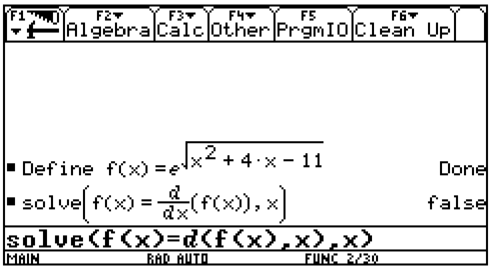
$x < -2 - \sqrt{15}$ $(\approx -5.87)$ or $x > -2 + \sqrt{15}$ $(\approx 1.87)$.




Since the graph passes through the point *P*(2, e) the domain interval must include *x* = 2 and the solution is:

$$y = \mathrm{e}^{+\sqrt{x^2+4x-11}},\ \ x > -2 + \sqrt{15}\ (\approx 1.87).$$

The last screen shows that the equation $f(x) = f'(x)$ has no solution for $x$. This can be seen in a different way, using PAP methods:

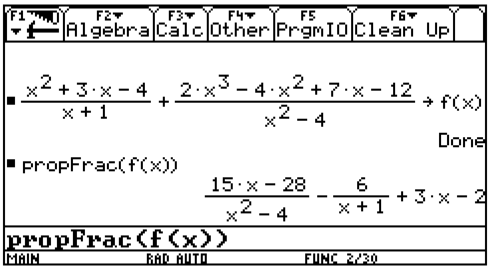



$$f(x) = f'(x) \Leftrightarrow f(x) = \frac{f(x)}{\ln(f(x))}\cdot(x+2)$$

$$\Leftrightarrow \ln(f(x)) = x + 2 \Leftrightarrow \sqrt{x^2 + 4x - 11} = x + 2$$

$$\Leftrightarrow x^2 + 4x - 11 = x^2 + 4x + 4 \Leftrightarrow -11 = 4.$$

**Question 7a (10%) [CAS version]** **Category C3, C4**




Consider the function

$$f(x) = \frac{x^2 + 3x - 4}{x+1} + \frac{2x^3 - 4x^2 + 7x - 12}{x^2 - 4}.$$

Find the domain of $f$.
Find an equation of each of the asymptotes.

**Answer to question 7a**

It is immediately apparent that the zeros of the denominators of $f$ are -1, -2, and 2. Thus the domain is $\mathbb{R} \setminus \{-1, -2, 2\}$.

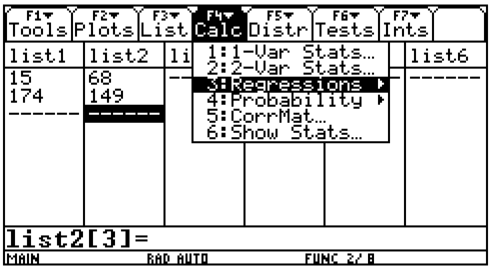
We expand the function into proper fractions. None of the three zeros of the denominators is also a zero of the numerator. Thus all three are vertical asymptotes.

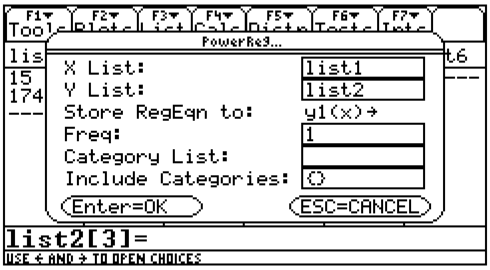
The two fractional parts of $f(x)$ both tend to zero as $x \to \pm\infty$, giving the equation $y = 3x - 2$ as the oblique asymptote.

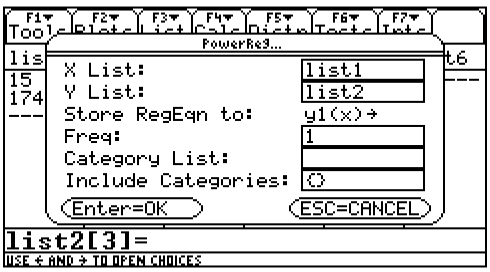
<u>Conclusion</u>

The graph has four asymptotes
$x = -2$, $x = -1$, $x = 2$, and $y = 3x - 2$.

**Question 7b (10% , no special CAS version)** **Category C0 (CAS neutral)**




For the Galapagos Islands, the number of species of terrestrial plants on any island can be determined with good approximation as a function $N(x) = b \cdot x^a$ of the area *x*, in square miles.

It is given that $N(15) = 68$ and $N(174) = 149$.

Calculate $a$ and $b$.

Consider two islands, where the area of one island is 2.5 times the area of the other island. Calculate the ratio between the number of terrestrial species on these two islands.

**Answer to question 7b**

The two given points are entered into lists, using the Stats/List Editor Application. By power regression (not all screens are shown in the right column) we find the function $N(x) \approx 28.6 \cdot x^{0.320}$.

By calculating $\frac{N(2.5 \cdot x)}{N(x)}$ we find that for a 2.5 times larger island the number of species must be 1.34 times bigger.